

# NEUTRINO OSCILLATIONS AND COSMOLOGY

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Phenomenology of neutrino oscillations in vacuum and in cosmological plasma is considered. Neutrino oscillations in vacuum are usually described in plane wave approximation. In this formalism there is an ambiguity if one should assume  $\delta p = 0$  and correspondingly  $\delta E \neq 0$  or vice versa  $\delta E = 0$  and  $\delta p \neq 0$ , or some other condition. We will use the standard set of quantum field theory rules and show that the theoretical description is unambiguous and that the state of oscillating neutrinos is determined by the conditions of their registration. The wave packet formalism is a natural result of such an approach.

Oscillating neutrinos in cosmological plasma cannot be described in terms of wave function because of a fast loss of coherence due to elastic or inelastic scattering, so that one should use the density matrix formalism. Kinetic equations for the density matrix of oscillating neutrinos are derived. Numerical and semi-analytical solutions of the equations are examined. In particular, a possibility of amplification of cosmological lepton asymmetry in the sector of active neutrinos mixed to sterile ones is critically discussed.

## 1 Neutrino oscillations in vacuum. Basic concepts.

As is very well known, neutrinos possibly oscillate if (and because) their mass eigenstates are different from their interaction eigenstates<sup>1-4</sup> (for a review and more references see<sup>5</sup>). In other words, the mass matrix of different neutrino species is not diagonal in the basis of neutrino flavors:  $[\nu_e, \nu_\mu, \nu_\tau]$ . The latter is determined by the interaction with charged leptons, so that a beam of e.g. electrons would produce  $\nu_e$  which is a mixture of several different mass states. An important condition is that the masses are different, otherwise oscillations would be unobservable. Indeed, if all the masses are equal, the mass matrix would be proportional to the unit matrix which is diagonal in any basis.

Of course not only neutrinos are capable to oscillate. All particles that are produced in the same reactions will do that, but usually the oscillation frequency,  $\omega_{osc} \sim \delta m^2/2E$  is so huge and correspondingly the oscillation length

$$l_{osc} = 2p/\delta m^2 \quad (1)$$

is so small, that the effect is very difficult to observe. Here  $E$  and  $p$  are

respectively the energy and momentum of the particles in consideration and  $\delta m^2 = m_1^2 - m_2^2$ . The expression is written in relativistic limit. Only for  $K$ -mesons and hopefully for neutrinos the mass difference is so small that  $l_{osc}$  is, or may be, macroscopically large.

The neutrino Lagrangian can be written as follows:

$$\mathcal{L}_\nu = i\bar{\nu}\not{\partial}\nu + \bar{\nu}\mathcal{M}\nu + \bar{\nu}\not{Z}\nu + \bar{\nu}\not{W}l \quad (2)$$

where the vector-column  $\nu = [\nu_e, \nu_\mu, \nu_\tau]^T$  is the operator of neutrino field in interaction basis,  $l = [e, \mu, \tau]^T$  is the vector of charged lepton operators; the last two terms describe respectively neutral and charge current interactions (with  $Z$  and  $W$  bosons). The upper index “T” means transposition. The matrix  $\mathcal{M}$  is the mass matrix and by assumption it is non-diagonal in the interaction basis. This is not necessary but quite natural because normally masses know nothing about interactions.

Transformation between mass and interaction eigenstates is realized by an orthogonal matrix  $U$  with the entries that are parameters of the theory to be determined by experiment. In the simplest case of only two mixed particles the matrix  $U$  has the form

$$U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (3)$$

If for example the only noticeable mixing is between electronic and muonic neutrinos, then the flavor eigenstates are related to the mass eigenstates  $\nu_{1,2}$  as:

$$\begin{aligned} \nu_e &= \nu_1 \cos \theta + \nu_2 \sin \theta, \\ \nu_\mu &= -\nu_1 \sin \theta + \nu_2 \cos \theta \end{aligned} \quad (4)$$

Thus if electronic neutrinos are produced on a target by a beam of electrons, their propagating wave function would have the form

$$\psi_{\nu_e}(\vec{r}, t) = \cos \theta |\nu_1\rangle e^{ik_1 x} + \sin \theta |\nu_2\rangle e^{ik_2 x} \quad (5)$$

where  $kx = \omega t - \vec{k}\vec{r}$  and sub- $\nu_e$  means that the initial state was pure electronic neutrino. Below (in this section only) we denote neutrino energy by  $\omega$  to distinguish it from the energies of heavy particles that are denoted as  $E$ . We assume, as is normally done, plane wave representation of the wave function.

If such a state hits a target, what is the probability of producing an electron or a muon? This probability is determined by the fraction of  $\nu_e$  and  $\nu_\mu$

components in the wave function  $\psi_{\nu_e}$  at space-time point  $x$ . The latter can be found by re-decomposition of  $\nu_{1,2}$  in terms of  $\nu_{e,\mu}$ :

$$\psi_{\nu_e}(\vec{r}, t) = \cos \theta e^{ip_1 x} (\cos \theta |\nu_e\rangle - \sin \theta |\nu_\mu\rangle) + \sin \theta e^{ip_2 x} (\sin \theta |\nu_e\rangle + \cos \theta |\nu_\mu\rangle) \quad (6)$$

One can easily find from that expression that the probability to register  $\nu_e$  or  $\nu_\mu$  are respectively:

$$P_{\nu_e}(\vec{r}, t) \sim \cos^4 \theta + \sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta \cos \delta \Phi, \quad (7)$$

$$P_{\nu_\mu}(\vec{r}, t) \sim 2 \sin^2 \theta \cos^2 \theta (1 - \cos \delta \Phi) \quad (8)$$

Here  $\delta \omega = \omega_1 - \omega_2$ ,  $\delta \vec{k} = \vec{k}_1 - \vec{k}_2$ , and

$$\delta \Phi(\vec{r}, t) = \delta \omega t - \delta \vec{k} \vec{r} \quad (9)$$

The energy difference between the mass eigenstates is

$$\delta \omega = \frac{\partial \omega}{\partial m^2} \delta m^2 + \frac{\partial \omega}{\partial \vec{k}} \delta \vec{k} \quad (10)$$

Using this expression we find for the phase difference

$$\delta \Phi(\vec{r}, t) = \frac{\delta m^2}{2\omega} t + \delta \vec{k} \left( \frac{\vec{k}}{\omega} t - \vec{r} \right) \quad (11)$$

The standard result of the neutrino oscillation theory are obtained if one assumes that: 1)  $\delta \vec{k} = 0$ , 2)  $\vec{k} = \omega \vec{v}$ , and 3)  $t = r/v$ :

$$\delta \Phi = \frac{\delta m^2 r}{2k} \quad (12)$$

Each of these assumptions is difficult to understand and, even more, some of them, in particular,  $\delta k = 0$  may be explicitly incorrect (see below). Both the second and the third conditions are fulfilled for a classical motion of a point-like body, however their validity should be questioned for a quantum mechanical particle (for a wave). Despite all that the final result (12) is true and if there are some corrections, they can be trivially understood.

Basic features of neutrino oscillations were discussed in many papers. An incomplete list of references includes <sup>6-25</sup>. One can find more citations and discussion in the above quoted papers and in the books <sup>26-30</sup>. Still some confusion and indications on possible controversies reappear from time to time

in the literature, so it seems worthwhile to present a consistent derivation of eq. (12) from the first principles. A large part of this section is based on the discussions (and unpublished work) with A.Yu. Morozov, L.B. Okun, and M.G. Schepkin.

Let us consider a localized source that produces oscillating neutrinos; we keep in mind, for example, a pion decaying through the channel  $\pi \rightarrow \mu + \nu_\mu$ . The wave function of the source  $\psi_s(\vec{r}, t)$  can be Fourier decomposed in terms of plane waves:

$$\begin{aligned} \psi_s(\vec{r}, t) &= \int d^3p \, C(\vec{p} - \vec{p}_0) e^{iEt - i\vec{p}\vec{r}} \approx \\ e^{iE_0t - i\vec{p}_0\vec{r}} \int d^3q \, C(\vec{q}) \exp \left[ -i\vec{q}(\vec{r} - \vec{V}_0t) \right] &= e^{iE_0t - i\vec{p}_0\vec{r}} \tilde{C}(\vec{r} - \vec{V}_0t) \end{aligned} \quad (13)$$

where  $\vec{V}_0 = \vec{p}_0/E_0$ . It is the standard wave packet representation. The function  $C(\vec{p} - \vec{p}_0)$  is assumed to be sharply peaked around the central momentum  $\vec{p}_0$  with dispersion  $\Delta\vec{p}$ . The particle is, by construction, on-mass-shell, i.e.  $E^2 = p^2 + m^2$ . This is of course also true for the central values  $E_0$  and  $p_0$ . As the last expression shows, the particle behaves as a plane wave with frequency and wave vector given respectively by  $E_0$  and  $\vec{p}_0$  and with the shape function (envelope) given by  $\tilde{C}(\vec{r} - \vec{V}_0t)$  which is the Fourier transform of  $C(\vec{q})$ . Evidently the envelope moves with the classical velocity  $\vec{V}_0$ . Characteristic size of the wave packet is  $l_{pack} \sim 1/\Delta p$ .

Let us consider the pion decay,  $\pi \rightarrow \mu + \nu$ . One would naturally expect  $\delta\omega \sim \delta k \sim \delta m^2/E$ . If this is true the probability of oscillation would be

$$P_{osc} \sim \cos \left[ \frac{x + b(x - Vt)}{l_{osc}} \right], \quad (14)$$

where  $l_{osc}$  is given by expression (1) and  $b$  is a numerical coefficient relating  $\delta p$  with  $l_{osc}$ . For simplicity the one-dimensional expression is presented.

Thus to obtain the probability of neutrino registration one should average the factor  $(x - Vt)$  over the size of the wave packet and for large packets, if  $b$  is non-negligible, a considerable suppression of oscillations should be expected. The size of the neutrino wave packet from the pion decay at rest is macroscopically large,  $l_{pack} \approx c\tau_\pi \approx 7.8$  m, where  $c$  is the speed of light and  $\tau_\pi = 2.6 \cdot 10^{-8}$  sec is the pion life-time. The oscillation length is  $l_{osc} = 0.4 \text{ m} (E/\text{MeV})/(\delta m^2/\text{eV}^2)$ , so  $l_{osc}$  could be smaller or comparable to  $l_{pack}$  and the effect of suppression of oscillations due to a finite size of the wave packet might be significant. It is indeed true but only for the decay of a moving pion, and this suppression is related to an uncertainty in the position

of decay. To see that we have to abandon the naive approach described above and to work formally using the standard set of quantum mechanical rules.

Let us assume that neutrinos are produced by a source with the wave function  $\psi_s(\vec{x}, t)$ . This source produces neutrinos together with some other particles. We assume first the following experimental conditions: neutrinos are detected at space-time point  $\vec{x}_\nu, t_\nu$ , while the accompanying particles are not registered. The complete set of stationary states of these particles is given by the wave functions  $\psi_n \sim \exp(iE_n t)$ . The amplitude of registration of propagating state of neutrino of type  $j$  (mass eigenstate) accompanying by other particles in the state  $\psi_n$  is given by

$$A_j^{(n)} = \int d\vec{r}_s dt_s \psi_s(\vec{r}_s, t_s) \psi_n(\vec{r}_s, t_s) G_{\nu_j}(\vec{r}_\nu - \vec{r}_s, t_\nu - t_s) \quad (15)$$

In principle one even does not need to know the concrete form of  $\psi_n$ , the only necessary property of these functions is the condition that they form a complete set:

$$\sum_n \psi_n(\vec{r}, t) \psi_n^*(\vec{r}', t) = \delta(\vec{r} - \vec{r}') \quad (16)$$

However in what follows we will use for simplicity the eigenfunctions of momentum,  $\psi_n \sim \exp(i\vec{p}\vec{r} - iEt)$ .

For the subsequent calculations we need the following representation of the Green's function which is obtained by the sequence of integration:

$$\begin{aligned} G(\vec{r}, t) &= \int \frac{d^4 p_4}{p_4^2 - m^2} e^{ip_4 x} = \\ &= 2\pi \int_{-\infty}^{+\infty} d\omega e^{i\omega t} \int_0^{+\infty} \frac{dp p^2}{\omega^2 - p^2 - m^2} \int_{-1}^1 d\zeta e^{-ipr\zeta} = \\ &= \frac{i\pi}{r} \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \int_{-\infty}^{+\infty} \frac{dp p^2}{\omega^2 - p^2 - m^2} (e^{ipr} - e^{-ipr}) \quad (17) \end{aligned}$$

Here  $\omega$  and  $p$  are respectively the fourth and space components of the 4-vector  $p_4$ . We have omitted spin matrices because the final result is essentially independent of that. The integration over  $dp$  was extended over the whole axis (from  $-\infty$  to  $+\infty$ ) because the integrand is an even function of  $p$ . This permit to calculate this integral by taking residues in the poles on mass shell:  $p = \pm\sqrt{\omega^2 - m^2 + i\epsilon}$ . Both poles give the same contribution, so skipping unnecessary numerical coefficients, we finally obtain:

$$G(\vec{r}, t) = \frac{1}{r} \int_{-\infty}^{+\infty} d\omega e^{-i\omega t + i\sqrt{\omega^2 - m^2} r} \quad (18)$$

As a source function  $\psi_s$  we will take essentially expression (13) but assume that the source is a decaying particle with the decay width  $\gamma$ , that was born at the moment  $t = 0$ . It corresponds to multiplication of  $\psi_s$  by  $\theta(t) \exp(-\gamma t)$ . Taking all together we obtain the following expression for the amplitude:

$$A_j^{(n)}(\vec{r}, t) = \int_0^\infty dt_s \int \frac{d\vec{r}_s}{|\vec{r} - \vec{r}_s|} \int d\vec{p} C(\vec{p} - \vec{p}_0) e^{iEt_s - i\vec{p}\vec{r}_s - \gamma t/2} e^{iE_n t_s + i\vec{p}_n \vec{r}_s} \int_{-\infty}^{+\infty} d\omega_j e^{i\omega_j(t-t_s) - ik_j|\vec{r} - \vec{r}_s|} \quad (19)$$

Integrals over  $d\vec{r}_s$  and  $d\vec{p}$  are taken over all infinite space. It is worthwhile to remind here that all the momenta are on mass shell,  $E^2 = p^2 + m_\pi^2$  (we assumed that the source is a decaying pion) and  $\omega_j^2 = k_j^2 + m_j^2$ , where  $m_j$  is the mass of  $j$ -th neutrino mass eigenstate.

The integration over  $dt_s$  is trivial and gives the factor  $(E - E_n - \omega_j + i\gamma/2)^{-1}$ . The integration over  $d\vec{r}_s$  can be easily done if the registration point is far from the source. In this case it is accurate enough to take  $1/|\vec{r} - \vec{r}_s| \approx 1/r$ , while the same quantity in the exponent should be expanded up to the first order:

$$|\vec{r} - \vec{r}_s| \approx r - \vec{\xi} \vec{r}_s \quad (20)$$

where  $\vec{\xi} = \vec{r}/r$  is a unit vector directed from the center of the source taken at the initial moment  $t = 0$  to the detector at the point  $\vec{r}$ . In this limit the integral over  $d\vec{r}_s$  gives  $\delta(\vec{p} - \vec{p}_n - \vec{k}_j)$ , ensuring momentum conservation:

$$\vec{p} = \vec{p}_n + \vec{k}_j \equiv \vec{p}_{\pi,j} \quad (21)$$

The vector of neutrino momentum is formally defined as

$$\vec{k}_j = \vec{\xi} k_j = \vec{\xi} \sqrt{\omega_j^2 - m_j^2} \quad (22)$$

Ultimately we are left with the integral:

$$A_j^{(n)} = \frac{1}{r} \int_{-\infty}^{+\infty} d\omega_j C(\vec{p}_n + \vec{k}_j - \vec{p}_0) \frac{e^{i\omega_j t - ik_j r}}{E_{\pi,j} - E_n - \omega_j + i\gamma/2} \quad (23)$$

where  $E_{\pi,j} = \sqrt{(\vec{p}_n + \vec{k}_j)^2 + m_\pi^2}$ . This integral can be taken in the 'pole approximation' and to do that we need to expand the integrand near the energy conservation law, see below eq. (26), as follows. The neutrino energy is presented as  $\omega_j = \omega_j^{(0)} + \Delta\omega_j$ . To avoid confusion one should distinguish

between the deviation of neutrino energy from the central value given by the conservation law,  $\Delta\omega_j$ , from the difference of energies of different neutrino mass eigenstates,  $\delta\omega = \omega_1 - \omega_2$ . The neutrino momentum is expanded up to the first order in  $\Delta\omega_j$ :

$$k_j = \sqrt{\omega_j^2 - m_j^2} \approx k_j^{(0)} + \Delta\omega_j/V_j^{(\nu)} \quad (24)$$

where  $V_j^{(\nu)} = k_j^{(0)}/\omega_j^{(0)}$  is the velocity of  $j$ -th neutrino. The pion energy is determined by the momentum conservation (21) and is given by

$$E_{\pi,j} = \sqrt{\left(\vec{p}_n + \vec{k}_j^{(0)} + \vec{\Delta}k_j\right)^2 + m_\pi^2} \approx E_{\pi,j}^{(0)} + \vec{V}_{\pi,j}\vec{\xi}\Delta\omega_j \quad (25)$$

where the pion velocity is  $\vec{V}_{\pi,j}^{(\pi)} = (\vec{p}_n + \vec{k}_j^{(0)})/E_{\pi,j}^{(0)}$ . The neutrino energy,  $\omega_j^{(0)}$ , satisfying the conservation law is defined from the equation:

$$E_{\pi,j}^{(0)} - E_n - \omega_j^{(0)} = 0 \quad (26)$$

Now the integral over  $\omega_j$  is reduced to

$$A_j^{(n)} = \frac{e^{i\omega_j^{(0)}t - ik_j^{(0)}r}}{r} C \left( \vec{p}_n + \vec{k}_j^{(0)} - \vec{p}_0 \right) \int_{-\infty}^{+\infty} d\Delta\omega_j \frac{e^{i\Delta\omega_j(t-r/V_j^{(\nu)})}}{\left( \vec{V}_{\pi,j}^{(\pi)}\vec{\xi}/V_j^{(\nu)} \right) \Delta\omega_j - \Delta\omega_j + i\gamma/2} \quad (27)$$

The last integral vanishes if  $t < V_j^{(\nu)}r$ , while in the opposite case it can be taken as the residue in the pole and we finally obtain:

$$A_j^{(n)} = \frac{C(\vec{p}_{\pi,j} - \vec{p}_0)}{r} \theta(r - V_j^{(\nu)}t) \exp \left( i\omega_j^{(0)}t - ik_j^{(0)}r - \frac{\gamma}{2} \frac{V_j^{(\nu)}t - r}{V_j^{(\nu)} - \vec{V}_{\pi,j}^{(\pi)}\vec{\xi}} \right) \quad (28)$$

We obtained a neutrino wave packet moving with the velocity  $V_j^{(\nu)}$  with a well defined front (given by the theta-function) and decaying with time in accordance with the decay law of the source. The similar wave packet, but moving with a slightly different velocity describes another oscillating state  $\nu_i$ . It is evident from this expressions that phenomenon of coherent oscillations takes place only if the packets overlap, as was noticed long ago<sup>6,8,9</sup>.

The probability of registration of oscillating neutrinos at the space-time point  $(\vec{r}, t)$  is given by the density matrix

$$\rho_{ij} = \int d\vec{p}_n A_i^{(n)}(\vec{r}_\nu, t_\nu) A_j^{*(n)}(\vec{r}_\nu, t_\nu) \quad (29)$$

The oscillating part of the probability is determined by the phase difference (9) but now the quantities  $\delta\omega$  and  $\delta k$  are unambiguously defined. To this end we will use the conservation laws (21,26). They give:

$$\delta\omega^{(0)} = \delta E_\pi \quad \text{and} \quad \delta\vec{k}^{(0)} = \delta\vec{p}_\pi \quad (30)$$

The variation of neutrino energy is given by

$$\delta\omega = V^{(\nu)}\delta k + \delta m^2/2\omega \quad (31)$$

while the variation of the pion energy can be found from expression (25):

$$\delta E^{(\pi)} = \vec{V}^{(\pi)}\delta\vec{k} \quad (32)$$

From these equations follows

$$\delta\omega = -\frac{\delta m^2}{2\omega} \frac{\vec{V}^{(\pi)}\vec{\xi}}{V^{(\nu)} - \vec{V}^{(\pi)}\vec{\xi}} \quad \text{and} \quad \delta k = -\frac{\delta m^2}{2\omega} \frac{1}{V^{(\nu)} - \vec{V}^{(\pi)}\vec{\xi}} \quad (33)$$

One sees that generally both  $\delta\omega$  and  $\delta k$  are non-vanishing. Only in the case of pion decay at rest,  $\delta\omega = 0$  but  $\delta k$  is non-zero in any case. Substituting the obtained results into expression (9) for the phase difference we come to the standard expression (12) if  $V_\pi = 0$ . This result shows a remarkable stability with respect to assumptions made in its derivation. However if the pion is moving, then the oscillation phase contains an extra term

$$\delta\Phi = \frac{\delta m^2}{2\omega} \frac{\vec{\xi}(\vec{r} - \vec{V}^{(\pi)}t)}{V^{(\nu)} - \vec{\xi}\vec{V}^{(\pi)}} = \frac{r\delta m^2}{2k} + \frac{(\vec{\xi}\vec{V}^{(\pi)})(r - V^{(\nu)}t)}{V^{(\nu)} - \vec{\xi}\vec{V}^{(\pi)}} \quad (34)$$

This extra term would lead to a suppression of oscillation after averaging over time. This suppression is related to the motion of the source and reflects the uncertainty in the position of pion at the moment of decay. So this result can be understood in the frameworks of the standard naive approach.

Similar expression can be derived for the case when both neutrino and muon from the decay  $\pi \rightarrow \mu + \nu_j$  are registered in the space-time points  $\vec{r}_\nu, t_\nu$  and  $\vec{r}_\mu, t_\mu$  respectively. This case was considered in refs. <sup>18,25</sup>. Here we will use the same approach as described above when the muon is not registered. The only difference is that in eq. (15) for the oscillation amplitude we have to substitute the Green's function of muon  $G_\mu(\vec{r}_\mu - r_s, t_\mu - t_s)$  instead of  $\psi_n(\vec{r}_s, t_s)$ . The calculations are essentially the same and after some algebra the following



expression for the oscillation amplitude is obtained:

$$A_{\mu,\nu} \sim \frac{V_\mu V_\nu}{r_\mu r_\nu} \Theta(L_\mu + L_\nu) \exp \left[ -\frac{\gamma(L_\mu + L_\nu)}{2(V_\mu + V_\nu)} \right] \tilde{C}(V_\mu L_\nu - V_\nu L_\mu) \exp \left[ i \left( k_\mu^{(0)} r_\mu + k_\nu^{(0)} r_\nu - E_\mu^{(0)} t_\mu - E_\nu^{(0)} t_\nu \right) \right] \quad (35)$$

where  $L = Vt - r$ . Each kinematic variable depends upon the neutrino state  $j$ , so they should contain sub-index  $j$ . The upper indices “0” mean that these momenta and energies are the central values of the corresponding wave packets, so that the classical relation  $\vec{k}^{(0)} = \vec{V}E^{(0)}$  holds for them. Here the direction of momenta are defined as above along the vector indicated to the observation point. However, the kinematics in this case is different from the previous one and the change of the energy and momentum of each particle for reactions with different sorts of neutrinos are related through the equations:

$$\delta E_\mu + \delta E_\nu = 0 \quad \text{and} \quad \delta \vec{k}_\mu + \delta \vec{k}_\nu = 0. \quad (36)$$

For the central values of momenta the following relations are evidently true,  $\delta k_\nu = \delta E_\nu - \delta m^2/2k_\nu$ , and the similar one for the muon without the last term proportional to the mass difference. Correspondingly the phase of oscillation is given by the expression

$$\delta \Phi = \delta k_\mu r_\mu - \delta E_\mu t_\mu + \delta k_\nu r_\nu - \delta E_\nu t_\nu = \delta E_\nu \left( \frac{r_\mu - V_\mu t_\mu}{V_\mu} - \frac{r_\nu - V_\nu t_\nu}{V_\nu} \right) - \frac{\delta m^2}{2k_\nu} r_\nu \quad (37)$$

The first two terms in the phase are proportional to the argument of  $\tilde{C}$  in eq. (35) and thus they give the contribution equal to the size of the the source, i.e. to the wave packet of the initial pion. If the latter is small (as usually the case) we obtain again the standard expression for the oscillation phase.

## 2 Matter effects

Despite extremely weak interactions of neutrinos, matter may have a significant influence on the oscillations if/because the mass difference between the propagating eigenstates is very small. Description of neutrino oscillations in matter was first done in ref. <sup>31</sup>. Somewhat later a very important effect of resonance neutrino conversion was discovered <sup>32</sup>, when even with a very small vacuum mixing angle, mixing in medium could reach the maximal value.

Hamiltonian of free neutrinos in the mass eigenstate basis has the form:

$$\mathcal{H}_m^{(1,2)} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \quad (38)$$

where  $E_j = \sqrt{p^2 + m_j^2}$ . In the interaction basis  $\mathcal{H}_m$  is rotated by the matrix (3):

$$\mathcal{H}_m^{(a,b)} = U \mathcal{H}_m^{(1,2)} U^{-1} = \begin{pmatrix} \cos^2 \theta E_1 + \sin^2 \theta E_2 & g \sin \theta \cos \theta \\ g \sin \theta \cos \theta & \sin^2 \theta E_1 + \cos^2 \theta E_2 \end{pmatrix} \quad (39)$$

Here  $g = \delta m^2 / 2E$  and we returned to the more usual notation  $E$  for neutrino energy.

The interaction Hamiltonian is diagonal in the interaction basis and if only first order effects in the Fermi coupling constant,  $G_F$ , are taken into account, then the Hamiltonian can be expressed through refraction index,  $n_a$ , of flavor  $a$ -neutrino in the medium (recall that the deviation of refraction index from unity is proportional to the forward scattering amplitude and thus contains  $G_F$  in the first power):

$$H_{int}^{(a,b)} = \begin{pmatrix} E(n_a - 1) & 0 \\ 0 & E(n_b - 1) \end{pmatrix} \quad (40)$$

where a small difference between  $E_1$  and  $E_2$  in front of small factors  $(n - 1)$  was neglected.

Thus, up to a unit matrix, the total Hamiltonian in the interaction basis can be written as

$$H_{tot}^{(a,b)} = \begin{pmatrix} f & g \sin 2\theta / 2 \\ g \sin 2\theta / 2 & 0 \end{pmatrix} \quad (41)$$

where  $f = g \cos 2\theta + E \delta n$  and  $\delta n = n_a - n_b$ . This matrix is easy to diagonalize. Its eigenvalues are

$$\lambda_{1,2} = \frac{f \pm \sqrt{f^2 + g^2 \sin^2 2\theta}}{2} \quad (42)$$

and the eigenstates in matter (up to normalization factor) are

$$|\nu_{1,2}\rangle = |\nu_a\rangle + \frac{g \sin 2\theta}{f \pm \sqrt{f^2 + g^2 \sin^2 2\theta}} |\nu_b\rangle \quad (43)$$

Refraction index may change with time, as happens in cosmology, or with space point, if neutrinos propagate in inhomogeneous medium, for example from the center of the Sun to its surface. If somewhere (or sometime)  $f$  vanishes then the resonance transition of one neutrino species to another is possible<sup>32</sup>. Indeed let us assume that  $\nu_e$  and  $\nu_\mu$  are mixed with a small vacuum mixing

angle  $\theta$  and that initially an electronic neutrino was produced in vacuum. So the initial propagating state would be mostly  $\nu_e$ :

$$|\nu_1\rangle_{in} = |\nu_e\rangle + (1/2) \tan 2\theta |\nu_\mu\rangle \quad (44)$$

After propagation in the media where the function  $f$  changes sign passing through zero, the propagating state would become mostly  $\nu_\mu$ :

$$|\nu_1\rangle_{fin} = |\nu_e\rangle - (2/\tan 2\theta) |\nu_\mu\rangle \quad (45)$$

This effect of resonance conversion may play an important role in the resolution of the solar neutrino problem and in cosmology.

### 3 Neutrino oscillations in cosmology

#### 3.1 A brief (and non-complete) review

Neutrino oscillations in the primeval plasma is significantly different from e.g. solar neutrino oscillations in the following two important aspects. First, cosmological plasma is almost charge symmetric. A relative excess of any particles over antiparticles is believed to be at the level  $10^{-9} - 10^{-10}$ , while in stellar material the asymmetry is of order unity. Neutrino oscillations in the early universe may change the magnitude of asymmetry in the sector of active neutrinos. On the other hand, the asymmetry has a strong influence on the oscillations through the refraction index of the primeval plasma (see below). It leads to a strong non-linearity of the problem and makes calculations quite complicated. Second important point is that neutrino mean free path in the early universe is quite small at high temperatures and hence breaking of coherence becomes essential. Because of that one cannot use wave functions for description of oscillations and should turn to the density matrix formalism. It leads to a great complexity of equations. Kinetic equations for density matrix with the account of neutrino scattering and annihilation were derived in the papers<sup>8,33,34,35</sup>. In ref.<sup>8</sup>, where the impact of neutrino oscillations on big bang nucleosynthesis (BBN) was first considered, only the second order effects, proportional to  $G_F^2$ , were taken into account, while the deviation of refraction index from unity was neglected. This approximation is valid for a sufficiently high  $\delta m^2$ . In a subsequent paper<sup>36</sup> implications for BBN of possible CP-violating effects in oscillations were discussed. Earlier works on neutrino oscillations also include refs.<sup>37,38,39</sup>. The calculations of refraction index in cosmological plasma were performed in ref.<sup>40</sup>. The results of this work permitted to make more accurate calculations of the role played by neutrino oscillations in BBN<sup>41–53</sup>.

It was noticed in ref.<sup>45</sup> that the oscillations between an active and sterile neutrinos could generate an exponential rise of lepton asymmetry in the sector

of active neutrinos. The origin of this instability is the following. Since lepton asymmetry comes with the opposite sign to refraction indices of neutrinos and antineutrinos (see below section 3.2), it may happen that the transformation of antineutrinos to their sterile partners would proceed faster than similar transformation of neutrinos, especially if resonance conditions are fulfilled. It would lead to an increase of the asymmetry and through the refraction index to more favorable conditions for its rise. However it was concluded<sup>45</sup> that the rise is not significant and the effect of the generated asymmetry on BBN is small. This conclusion was reconsidered in ref.<sup>54</sup> where it was argued that asymmetry generated by this mechanism could reach very large values, close to unity, and this effect, in accordance to the earlier paper of the same group<sup>55</sup>, would have a significant influence on primordial abundances. This result attracted a great attention and was confirmed in several subsequent publications<sup>56–61</sup>. Moreover, some works showed not only rising and large asymmetry but also a chaotic behavior of its sign<sup>62–66</sup>.

Due to complexity of equations some simplifying approximations were made in their solutions. First accurate numerical solution of (almost) exact equations were done in refs.<sup>67–71</sup>. The most essential approximation was that the lost of coherence was described by the term  $\gamma(\rho_{eq} - \rho)$  instead of the exact two-dimensional collision integral taken from the square of the scattering amplitude over momenta. In fact this approximation was used in all the papers. The calculations of refs.<sup>67–71</sup> were performed for rather small mass difference,  $\delta m^2 < 10^{-7} \text{ eV}^2$ . Neither chaoticity, nor a considerable rise of the asymmetry were found. For a larger mass difference a strong numerical instability was observed. However this result is not in a contradiction with other papers because the latter found the above mentioned effects for much larger mass differences. In our work<sup>72</sup> we tried to extend the range of validity of direct numerical calculations to large  $\delta m^2$ . We were able to proceed only up to  $\delta m^2 \approx 10^{-6}$ . For higher values we did not find a way to avoid numerical instability. To overcome the problem we analytically transformed kinetic equations to a simpler form that permitted a stable numerical integration. To this end an expansion in terms of a large parameter that corresponds to a high frequency of oscillations was used. This method is rather similar to the well known separation of fast and slow variables in differential equations. According to our results the asymmetry may rise, but only by 5-6 orders of magnitude (from initial  $10^{-10}$ ), i.e. 4-5 orders of magnitude weaker than was obtained in other papers<sup>54,56,57,58,59,60,61</sup>. However in agreement with these papers no chaoticity was found. Our results were criticized in ref.<sup>66</sup> where it was argued that lepton asymmetry generated by oscillations must be chaotic. However the author misunderstood our calculations and criticized us for the things that we

never did and, second, the approximation used in that paper<sup>66</sup>, namely calculations in terms of one fixed value of neutrino momentum that was chosen to be equal to the thermally averaged one, is intrinsically inappropriate for the solution of the problem of chaoticity. In that approximation even neutrino oscillations in vacuum would result in chaotic lepton asymmetry.

Thus, most probably a chaotic amplification of the lepton asymmetry does not take place, however the exact magnitude of the amplification remains uncertain. As discussed in ref.<sup>58,61</sup> their recent calculations are exact and do not suffer from numerical instability. On the other hand, we do not see any shortcomings of our semi-analytical approach<sup>72</sup>, all approximations are well under control and numerical part of calculations is quite simple and stable. More work is necessary to resolve the contradictions both in magnitude and possible chaoticity.

### 3.2 Refraction index

In this section we derive the Schroedinger equation for neutrino wave function in the primeval plasma. We will start with the neutrino quantum operator  $\nu_a(x)$  of flavor  $a$  that satisfies the usual Heisenberg equation of motion:

$$(i\partial - \mathcal{M}) \nu_a(x) + \frac{g}{2\sqrt{2}} \delta_{ae} W_\alpha(x) O_\alpha^{(+)} e(x) + \frac{g}{4 \cos \theta_W} Z_\alpha(x) O_\alpha^{(+)} \nu_a(x) = 0 \quad (46)$$

where  $\mathcal{M}$  is the neutrino mass matrix,  $W(x)$ ,  $Z(x)$  and  $e(x)$  are respectively the quantum operators of intermediate bosons and electrons, and  $O_\alpha^\pm = \gamma_\alpha (1 \pm \gamma_5)$ . We assumed that the temperature of the plasma is in MeV range and thus only electrons are present in the plasma.

Equations of motion for the field operators of  $W$  and  $Z$  bosons have the form

$$G_{W,\alpha\beta}^{-1} W_\beta(x) = \frac{g}{2\sqrt{2}} \bar{\nu}_a(x) O_\alpha^{(+)} \nu_a(x), \quad (47)$$

$$G_{Z,\alpha\beta}^{-1} Z_\beta(x) = \frac{g}{4 \cos \theta_W} \left[ \bar{\nu}_a(x) O_\alpha^{(+)} \nu_a(x) + (2 \sin^2 \theta_W - 1) \bar{e}(x) O_\alpha^{(+)} e(x) + 2 \sin^2 \theta_W \bar{e}(x) O_\alpha^{(-)} e(x) \right] \quad (48)$$

where the differential operators  $G_{W,Z}^{-1}$  are inverse Green's functions of  $W$  and  $Z$ . In momentum representation they can be written as

$$G_{\alpha\beta} = \frac{g_{\alpha\beta} - q_\alpha q_\beta / m^2}{m^2 - q^2} \quad (49)$$

It can be shown that the term  $q_\alpha q_\beta / m^2$  gives contribution proportional to lepton masses and can be neglected.

In the limit of small momenta,  $q \ll m_{W,Z}$ , equation (47) can be solved as

$$W_\alpha(x) = -\frac{g}{2\sqrt{2}m_W^2} \left(1 - \frac{\partial^2}{m_W^2}\right) \left(\bar{e}(x)O_\alpha^{(+)}\nu_e(x)\right) \quad (50)$$

A similar expression with an evident substitution for the r.h.s. can be obtained for  $Z_\alpha(x)$ . These expressions should be substituted into eq. (46) to obtain equation that contains only field operators of leptons,  $\nu_a(x)$  and  $e(x)$ :

$$\begin{aligned} (i\partial - \mathcal{M})\nu_a(x) = & \frac{G_F}{\sqrt{2}} \left\{ \delta_{ae} \left[ \left(1 - \frac{\partial^2}{m_W^2}\right) \left(\bar{e}(x)O_\alpha^{(+)}\nu_e(x)\right) \right] O_\alpha^{(+)}e(x) + \right. \\ & \frac{1}{2} \left[ \left(1 - \frac{\partial^2}{m_Z^2}\right) \left(\bar{\nu}_b(x)O_\alpha^{(+)}\nu_b(x) + (2\sin^2\theta_W - 1)\bar{e}(x)O_\alpha^{(+)}e(x) + \right. \right. \\ & \left. \left. 2\sin^2\theta_W \bar{e}(x)O_\alpha^{(-)}e(x)\right) \right] O_\alpha^{(+)}\nu_a(x) \left. \right\} \quad (51) \end{aligned}$$

Neutrino wave function in the medium is defined as

$$\Psi_a(x) = \langle A | \nu_a(x) | A + \nu^{(k)} \rangle \quad (52)$$

where  $A$  describes the state of the medium and  $\nu^{(k)}$  is a certain one-neutrino state, specified by quantum numbers  $k$ , e.g. neutrino with momentum  $\vec{k}$ . Equation of motion for this wave function can be found from eq. (51) after averaging over medium. The theory is quantized perturbatively in the standard way. We define the free neutrino operator  $\nu_a^{(0)}$  that satisfies the equation of motion:

$$(i\partial - \mathcal{M})\nu_a^{(0)}(x) = 0 \quad (53)$$

This operator is expanded as usually in terms of creation-annihilation operators:

$$\nu^{(0)}(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} \sum_s \left( a_k^s u^s(k) e^{ikx} + b_k^{s\dagger} v^s(k) e^{ikx} \right) \quad (54)$$

and one-particle state is defined as  $|\nu^{(k)}\rangle = a_k^\dagger |\text{vac}\rangle$ .

The equation of motion for the neutrino wave function  $\Psi_a(x)$  can be obtained from expression (52) perturbatively by applying the operator  $(i\partial - \mathcal{M})$  and using eq. (51) with free neutrino operators  $\nu_a^{(0)}$  in the r.h.s. After some algebra which mostly consists in using equations of motion for the free fermion

operators and (anti)commutation relations between the creation/annihilation operators, one would obtain the equation of the form:

$$i\partial_t\Psi(t) = (\mathcal{H}_m + V_{eff})\Psi \quad (55)$$

where  $\mathcal{H}_m$  is the free Hamiltonian; in the mass eigenstate basis it has the form  $\mathcal{H}_0 = \text{diag} \left[ \sqrt{p^2 + m_j^2} \right]$ . The matrix-potential  $V_{eff}$  describes interactions of neutrinos with media and is diagonal in the flavor basis. Up to the factor  $E$  (i.e. neutrino energy) it is essentially the refraction index of neutrino in the medium. The potential contains two terms. The first one comes from the averaging of the external current  $J \sim \bar{l}O_\alpha l$ . Due to homogeneity and isotropy of the plasma only its time component is non-vanishing and proportional to the charge asymmetry (i.e. to the excess of particles over antiparticles) in the plasma. This term has different signs for neutrinos and antineutrinos. However the interactions of neutrinos with the medium is not always of the (current) $\times$ (current) form due to non-locality related to the exchange of  $W$  or  $Z$  bosons. If incoming and outgoing neutrinos interact in different space-time points, the interaction with the medium cannot be written as an interaction with the external current. The contribution of such terms is inversely proportional to  $m_{W,Z}^2$  but formally it is of the first order in  $G_F$ . With these two types of contributions the diagonal matrix elements of the effective potential for the neutrino of flavor  $a$  has the form:

$$V_{eff}^a = \pm C_1 \eta G_F T^3 + C_2^a \frac{G_F^2 T^4 E}{\alpha}, \quad (56)$$

where  $E$  is the neutrino energy,  $T$  is the temperature of the plasma,  $G_F = 1.166 \cdot 10^{-5} \text{ GeV}^{-2}$  is the Fermi coupling constant,  $\alpha = 1/137$  is the fine structure constant, and the signs “ $\pm$ ” refer to anti-neutrinos and neutrinos respectively (this choice of sign describes the helicity state, negative for  $\nu$  and positive for  $\bar{\nu}$ ). According to ref. <sup>40</sup> the coefficients  $C_j$  are:  $C_1 \approx 0.95$ ,  $C_2^e \approx 0.61$  and  $C_2^{\mu,\tau} \approx 0.17$ . These values are true in the limit of thermal equilibrium, but otherwise these coefficients are some integrals from the distribution functions over momenta. For oscillating neutrinos deviations from thermal equilibrium could be significant and in this case expression (56) should be modified. However it is technically rather difficult to take this effect into account in numerical calculations and the simplified version (56) is used.

### 3.3 Loss of Coherence and Density Matrix

Breaking of coherence appears in the second order in the Fermi coupling constant  $G_F$ , so that equations of motion for the operators of *all* leptonic fields

(including electrons) should be solved up to the second order in  $G_F$ . Since the calculations are quite lengthy, we only sketch the derivation here. In the considered approximation the lepton operators  $l(x)$ , where  $l$  stands for neutrino or electron, in the r.h.s. of eq. (51) should be expanded up to the first order in  $G_F$ . The corresponding expressions can be obtained from the formal solution of eq. (51) up to first order in  $G_F$ . Their typical form is the following:

$$l = l_0 + G_l * (\text{r.h.s.}_0) \quad (57)$$

where the matrix (in neutrino space)  $G_l$  is the Green's function of the corresponding lepton and  $\text{r.h.s.}_0$  is the right hand side of eq. (51) taken in the lowest order in  $G_F$ , i.e with lepton operators taken in the zeroth order,  $l = l_0$ . Expression (57) should be substituted back into eq. (51) and this defines the r.h.s. up to the second order in  $G_F$  in terms of free lepton operators  $l_0$ . Of course in the second approximation we neglect the non-local terms,  $\sim 1/m_{W,Z}^2$ .

Now we can derive kinetic equation for the density matrix of neutrinos,  $\hat{\rho}_j^i = \nu^i \nu_j^*$ , where over-hat indicates that  $\hat{\rho}$  is a quantum operator. The  $C$ -valued density matrix is obtained from it by taking matrix element over the medium,  $\rho = \langle \hat{\rho} \rangle$ . We should apply to it the differential operator  $(i\partial - \mathcal{M})$  and use eq. (51). The calculations of matrix elements of the free lepton operators  $l_0$  are straightforward and can be achieved by using the standard commutation relations. There is an important difference between equations for the density matrix and the wave function. The latter contains only terms proportional to the wave function,  $i\partial_t \Psi = \mathcal{H}\Psi$ , while equation for density matrix contains source term that does not vanish when  $\rho = 0$ . Neutrino production or destruction is described by the imaginary part of the effective Hamiltonian. The latter is not hermitian because the system is not closed. By the optical theorem the imaginary part of the Hamiltonian is expressed through the cross-section of neutrino creation or annihilation. Such terms in kinetic equation for the density matrix are similar to the “normal” kinetic equation for the distribution functions:

$$\frac{df_1}{dt} = I_{coll} \quad (58)$$

where the collision integral is the integral over the proper phase space from the following combination of the distribution functions

$$F = -f_1 f_2 (1 - f_3)(1 - f_4) + f_4 f_3 (1 - f_1)(1 - f_2) \quad (59)$$

A similar combination appears for the case of oscillating neutrinos but with a rather complicated matrix structure. For example there can be the contribution of the form of the anti-commutator:

$$- \{ \rho_1, g(1 - \rho_3)g\rho_2(1 - \rho_4)g \} + (\text{inverse reaction}) \quad (60)$$



and a few more of different structure that results in a very lengthy expression. The matrix  $g$  describes neutrino interactions. It is diagonal in the flavor basis and has entries proportional to matrix elements squared of the relevant reactions. The Fermi-blocking factors  $(1 - \rho)$  appear when one takes matrix elements of neutrino operators over the medium in which neutrino occupation numbers may be non-zero.

We will not present here the complete form of the equation. It can be found e.g. in the paper<sup>35</sup>. Moreover in all the applications a “poor man” substitution has been done: all terms describing neutrino production or destruction were mimicked by

$$- \Gamma (\rho - \rho_{eq}) \quad (61)$$

where  $\rho_{eq}$  is the equilibrium value of the density matrix, i.e. the unit matrix multiplied by the equilibrium distribution function

$$f_{eq} = [\exp(E/T - \xi) + 1] \quad (62)$$

and the coefficient  $\Gamma$  is the reaction rate (see below).

### 3.4 Oscillations and lepton asymmetry

As we have already mentioned oscillations between active and sterile neutrinos may induce a significant lepton asymmetry in the sector of active neutrinos. Now we will consider this phenomenon in some more detail. Basic equations governing the evolution of the density matrix are:

$$i(\partial_t - Hp\partial_p)\rho_{aa} = F_0(\rho_{sa} - \rho_{as})/2 - i\Gamma_0(\rho_{aa} - f_{eq}) , \quad (63)$$

$$i(\partial_t - Hp\partial_p)\rho_{ss} = -F_0(\rho_{sa} - \rho_{as})/2 , \quad (64)$$

$$i(\partial_t - Hp\partial_p)\rho_{as} = W_0\rho_{as} + F_0(\rho_{ss} - \rho_{aa})/2 - i\Gamma_1\rho_{as} , \quad (65)$$

$$i(\partial_t - Hp\partial_p)\rho_{sa} = -W_0\rho_{sa} - F_0(\rho_{ss} - \rho_{aa})/2 - i\Gamma_1\rho_{sa} , \quad (66)$$

where  $a$  and  $s$  mean “active” and “sterile” respectively,  $F_0 = \delta m^2 \sin 2\theta/2E$ ,  $W_0 = \delta m^2 \cos 2\theta/2E + V_{eff}^a$ ,  $H = \sqrt{8\pi\rho_{tot}/3M_p^2}$  is the Hubble parameter,  $p$  is the neutrino momentum. The antineutrino density matrix satisfies the similar set of equations with the opposite sign of the antisymmetric term in  $V_{eff}^a$  and with a slightly different damping factor  $\bar{\gamma}$  (this difference is proportional to the lepton asymmetry in the primeval plasma).

Equations (63-66) account exactly for the first order terms described by the refraction index, while the second order terms describing breaking of coherence

are approximately modeled by the damping coefficients  $\Gamma_j$  in accordance with eq. (61). The latter are equal to<sup>73</sup>:

$$\Gamma_0 = 2\Gamma_1 = g_a \frac{180\zeta(3)}{7\pi^4} G_F^2 T^4 p . \quad (67)$$

In general the coefficient  $g_a(p)$  is a momentum-dependent function, but in the approximation of neglecting  $[1 - f]$  factors in the collision integral it becomes a constant<sup>74</sup> equal respectively to  $g_{\nu_e} \simeq 4$  and  $g_{\nu_\mu, \mu_\tau} \simeq 2.9$ <sup>47</sup>. In the following we will use more accurate values found from the thermal average of the complete electro-weak rates (with factors  $[1 - f]$  included), which we calculated numerically from our Standard Model code<sup>18</sup>. This gives us  $g_{\nu_e} \simeq 3.56$  and  $g_{\nu_\mu, \mu_\tau} \simeq 2.5$ .

It is convenient to introduce new variables

$$x = m_0 R(t) \text{ and } y = p R(t) \quad (68)$$

where  $R(t)$  is the cosmological scale factor so that  $H = \dot{R}/R$  and  $m_0$  is an arbitrary mass (just normalization), we choose  $m_0 = 1$  MeV. In the approximation that we will work, we assume that  $\dot{T} = -HT$ , so that we can take  $R = 1/T$ . In terms of these variables the differential operator  $(\partial_t - Hp\partial_p)$  transforms to  $Hx\partial_x$ . We will normalize the density matrix elements to the equilibrium function  $f_{eq}$ :

$$\rho_{aa} = f_{eq}(y)[1 + a(x, y)], \quad \rho_{ss} = f_{eq}(y)[1 + s(x, y)] , \quad (69)$$

$$\rho_{as} = \rho_{sa}^* = f_{eq}(y)[h(x, y) + il(x, y)] , \quad (70)$$

and express the neutrino mass difference  $\delta m^2$  in  $\text{eV}^2$ .

As the next step we will take the sum and difference of eqs. (63)-(66) for  $\nu$  and  $\bar{\nu}$ . The corresponding equations have the following form:

$$s'_\pm = Fl_\pm , \quad (71)$$

$$a'_\pm = -Fl_\pm - 2\gamma_+ a_\pm - 2\gamma_- a_\mp , \quad (72)$$

$$h'_\pm = Ul_\pm - VZl_\mp - \gamma_+ h_\pm - \gamma_- h_\mp , \quad (73)$$

$$l'_\pm = \frac{F}{2}(a_\pm - s_\pm) - Uh_\pm + VZh_\mp - \gamma_+ l_\pm - \gamma_- l_\mp , \quad (74)$$

where  $a_\pm = (a \pm \bar{a})/2$  etc, and the prime means differentiation with respect to  $x$ . We have used  $W = U \pm VZ$ ,  $\gamma = \Gamma_1/Hx$ , and  $\gamma_\pm = (\gamma \pm \bar{\gamma})/2$ , where  $\gamma_-$  parameterizes the difference of interaction rates between neutrino and anti-neutrinos, which is proportional to the neutrino asymmetry. With the approx-

imation  $\rho_{tot} \simeq 10.75\pi^2 T^4/30$ , the expressions for  $U$ ,  $V$ , and  $Z$  become:

$$U = 1.12 \cdot 10^9 \cos 2\theta \delta m^2 \frac{x^2}{y} + 26.2 \frac{y}{x^4}, \quad (75)$$

$$V = \frac{29.6}{x^2}, \quad (76)$$

$$Z = 10^{10} \left( \eta_o - \int \frac{dy}{4\pi^2} y^2 f_{eq} a_- \right), \quad (77)$$

where  $\eta_o$  is the asymmetry of the other particle species:

$$\eta_o^e = 2\eta_{\nu_e} + \eta_{\nu_\mu} + \eta_{\nu_\tau} + \eta_e - \eta_n/2 \quad (\text{for } \nu_e), \quad (78)$$

$$\eta_o^\mu = 2\eta_{\nu_\mu} + \eta_{\nu_e} + \eta_{\nu_\tau} - \eta_n/2 \quad (\text{for } \nu_\mu), \quad (79)$$

and  $\eta$  for  $\nu_\tau$  is obtained from eq. (79) by the interchange  $\mu \leftrightarrow \tau$ . The asymmetry is normalized in the same way as the neutrino asymmetry (the second term in (77)). Here we have implicitly assumed that  $\nu_a = \nu_e$ .

Up to this point our equations are essentially the same as those used by other groups. The equations look rather innocent and at first sight one does not anticipate any problem with their numerical solution. However the contribution from  $Z$  could be quite large with the increasing magnitude of the asymmetry. The exact value of the latter is determined by a delicate cancellation of the contributions from all energy spectrum of neutrinos. The function  $a_-$  under momentum integral is quickly oscillating and very good precision is necessary to calculate the integral with a desired accuracy. Even a small numerical error results in a large instability. To avoid this difficulty we analytically separated fast and slow variables in the problem and reduced this set of equations to a single differential equation for the asymmetry that can be easily numerically integrated. The corresponding algebra is somewhat complicated and we will not discuss it here. One can find the details in ref.<sup>72</sup>.

We have found that asymmetry practically does not rise for a large mixing angles,  $\sin \theta > 0.01$ . For smaller mixings some rise of the asymmetry is observed, though much weaker than that obtained in refs.<sup>54,56,57,58,59,60,61</sup>. For example for  $\delta m^2 = 1 \text{ eV}^2$  we found that asymmetry rises by approximately 5 orders of magnitude reaching the value around  $10^{-5}$  for  $\sin 2\theta$  in the interval  $10^{-5} - 10^{-3}$ . For  $\delta m^2 = 10^6 \text{ eV}^2$  the symmetry could rise up to 0.01 for  $\sin 2\theta = 3 \cdot 10^{-6} - 3 \cdot 10^{-5}$ , and only for huge mass difference  $\delta m^2 = 10^9$  the asymmetry may reach unity in a rather narrow range of mixing angles,  $\sin 2\theta = 2 \cdot 10^{-6} - 4 \cdot 10^{-6}$ . For  $\sin 2\theta$  outside the indicated limits the asymmetry does not rise. For even larger mass difference,  $\delta m^2 = 10^{12} \text{ eV}^2$ , the asymmetry practically does not rise.

Resolution of the contradiction between different groups is very important for the derivation of the constraints on the parameters of the oscillations from big bang nucleosynthesis (BBN). There are several effects by which the oscillations may influence abundances of light elements:

1. If sterile neutrinos are excited by the oscillations then the effective number of neutrino species at nucleosynthesis would be larger than 3. This effect, as is well known, results in an increase of mass fraction of helium-4 and deuterium.
2. Oscillations may distort the spectrum of neutrinos and, in particular, of electronic neutrinos. The sign of the effect is different, depending on the form of spectral distortion. A deficit of electronic neutrinos at high energy results in a smaller mass fraction of helium-4, while a deficit of  $\nu_e$  at low energy works in the opposite direction. A decrease of total number/energy density of  $\nu_e$  would result in an earlier freezing of neutrino-to-proton ratio and in a larger fraction of helium-4.
3. Oscillations may create an asymmetry between  $\nu_e$  and anti- $\nu_e$ . The  $n/p$  ratio in this case would change as  $n/p \sim \exp(\mu/T)$ , where  $\mu$  is the chemical potential corresponding to the asymmetry. With the present day accuracy of the data the asymmetry in the sector of electronic neutrinos could be at the level of a few per cent, i.e. much larger than the standard  $10^{-10}$ . Even if asymmetry is strongly amplified but still remains below 0.01 its direct influence on BBN would be negligible. It may however have an impact on the nucleosynthesis in an indirect way. Namely, the rise of the asymmetry by several orders of magnitude could suppress neutrino oscillations through refraction index so that new neutrino species corresponding to sterile neutrinos are not efficiently excited.

Thus one sees that the effects of oscillations may result both in a reduction or in an increase the effective number of neutrino species. In particular, oscillations may open room for additional particles at BBN. This conclusion strongly depends upon the magnitude of lepton asymmetry generated by the oscillations. Thus a resolution of the controversies between different theoretical calculations would be very important.

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## References

1. B. Pontecorvo, *ZhETF* **33** (1957) 549; English translation *JETP*, **6** (1958) 429; *ZhETF* **34** (1958) 247.
2. Z. Maki, M. Nakagawa, and S. Sakata, *Progr. Theor. Phys.*, **28** (1962) 870.
3. V. Gribov and B. Pontecorvo, *Phys. Lett.*, **28B** (1969) 493.
4. J.N. Bahcall and S. Frautschi, *Phys. Lett.*, **29B** (1969) 623.
5. S.M. Bilenky and B. Pontecorvo, *Phys. Repts.*, **41C** (1978) 225.
6. S. Nussinov, *Phys. Lett. B* **63** (1976) 201.
7. R.G. Winter, *Lett. al Nuovo Cimento*, **30** (1981) 101.
8. A.D. Dolgov, *Yad.Fiz. (Sov. J. Nucl. Phys.)*, **33** (1981) 1309.
9. B. Kayser, *Phys. Rev. D* **24** (1981) 110.
10. C. Giunti, C.W. Kim, and U.W. Lee *Phys. Rev. D*, **44** (1991) 3635; *Phys. Rev. D*, **45** (1992) 2414.
11. A. Yu. Smirnov and G.T. Zatspepin, *Mod. Phys. Lett*, **7** (1992) 1273.
12. J. Rich, *Phys. Rev. D*, **48** (1993) 4318.
13. M.M. Nieto, *Hyperfine Interact.* **100** (1996) 193.
14. T. Goldman, hep-ph/9604357.
15. Y. Grossman and H. J. Lipkin, *Phys. Rev. D*, **55** (1997) 2760.
16. D. V. Ahluwalia and T. Goldman, *Phys. Rev. D*, **56** (1997) 1698.
17. J.E. Campagne, *Phys. Lett.*, **B400** (1997) 135.
18. A.D. Dolgov, A.Yu. Morozov, L.B. Okun, and M.G. Schepkin, *Nucl. Phys.* **502** (1997) 3.
19. B. Kayser, hep-ph/9702327.
20. Yu. Shtanov, *Phys.Rev.* **D57** (1998) 4418
21. L. Stodolsky, *Phys. Rev. D*, **58** (1998) 036006.
22. H. Lipkin, hep-ph/9901399.
23. P. Fisher, B. Kayser, and K.S. McFarland, *Ann. Rev. Nucl. Part. Sci.* **49** (1999) 481.
24. A. Ioannisian and A. Pilaftsis, *Phys.Rev.* **D59** (1999) 053003.
25. M. Nauenberg, *Phys.Lett.*, **B447** (1999) 23.
26. J.N. Bahcall, *Neutrino Astrophysics*. Cambridge University Press, 1989.
27. R.N. Mohapatra and P.B. Pal, *Massive Neutrinos in Physics and Astrophysics*, World Scientific, 1991.
28. F. Boehm and P. Vogel, *Physics of Massive Neutrinos*, Cambridge University Press, 1992.
29. C.W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics*, Harwood Academic Press, 1993.
30. M. Fukugita and T. Yanagita, *Physics and Astrophysics of Neutrinos*,

Springer-Verlag, Tokyo, 1994.

31. L. Wolfenstein, *Phys. Rev.* **D17** (1978) 2369.
32. S.P. Mikheev and A.Yu. Smirnov, *Yad. Fiz.*, **42** (1985) 1441; *Nuov. Cim.*, **9C** (1986) 17.
33. L. Stodolsky, *Phys. Rev.* **D36** (1987) 2273.
34. G. Raffelt, G. Sigl, and L. Stodolsky, *Phys. Rev. Lett.* **70** (1993) 2363.
35. G. Sigl and G. Raffelt, *Nucl. Phys.* **B406** (1993) 423.
36. M.Yu. Khlopov and S. Petkov, *Phys. Lett.* **B99** (1981) 117, (E) **B100** (1981) 520.
37. D. Fargion and M. Shepkin, *Phys. Lett.* **B146** (1984) 46.
38. P.G. Langacker, B. Sathiadalan, and G. Steigman, *Nucl. Phys.* **B266** (1986) 669.
39. P. Langacker, S. Petcov, G. Steigman, and S. Toshev, *Nucl. Phys.* **282** (1987) 589.
40. D. Nötzold and G. Raffelt, *Nucl. Phys.* **B307** (1988) 924.
41. R. Barbieri and A. Dolgov *Phys. Lett.* **B237** (1990) 440.
42. K. Enqvist, K. Kainulainen and J. Maalampi, *Phys. Lett.* **B244** (1990) 186.
43. K. Kainulainen, *Phys. Lett.* **B244** (1990) 191.
44. K. Enqvist, K. Kainulainen and J. Maalampi, *Phys. Lett.* **B249** (1990) 531.
45. R. Barbieri and A. Dolgov, *Nucl. Phys.* **B237** (1991) 742.
46. K. Enqvist, K. Kainulainen and J. Maalampi, *Nucl. Phys.* **B349** (1991) 754.
47. K. Enqvist, K. Kainulainen and M. Thomson, *Phys. Lett.* **B280** (1992) 245.
48. K. Enqvist, K. Kainulainen and M. Thomson, *Nucl. Phys.* **B373** (1992) 498.
49. J. Cline, *Phys. Rev. Lett.* **68** (1992) 3137.
50. X. Shi, D. Schramm and B. Fields, *Phys. Rev.* **D48** (1993) 2563.
51. B.H.J. McKellar and M.J. Thomson, *Phys. Rev.* **D49** (1994) 2710.
52. V. A. Kostelecky and S. Samuel, *Phys. Rev.* **D52** (1995) 3184.
53. V. A. Kostelecky and S. Samuel, *Phys. Lett.* **B 385** (1996) 159.
54. R. Foot, M. Thomson and R.R. Volkas, *Phys. Rev.* **D53** (1996) 5349.
55. R. Foot, R.R. Volkas, *Phys. Rev. Lett.* **75** (1995) 4350.
56. R. Foot and R.R. Volkas, *Phys. Rev.* **D55** (1997) 5147.
57. N. Bell, R. Foot, and R. Volkas, *Phys. Rev.*, **D58** (1998) 105010.
58. R. Foot, *Astropart. Phys.* **10** (1999) 253.
59. P. Di Bari, P. Lipari, and M. Lusignoli, hep-ph/9907548.
60. P. Di Bari, hep-ph/9911214.

- 61. P. Di Bari, R. Foot, hep-ph/9912215.
- 62. X. Shi, *Phys. Rev.* **D54** (1996) 2753.
- 63. X. Shi, G.M. Fuller, *Phys.Rev.Lett.*, **83** (1999) 3120.
- 64. X. Shi, G.M. Fuller, *Phys.Rev.*, **D60** (1999) 063002.
- 65. K. Enqvist, K. Kainulainen, and A. Sorri, *Phys.Lett.* B464 (1999) 199.
- 66. A. Sorri, hep-ph/9911366.
- 67. D.P.Kirilova and M.V.Chizhov, *Phys.Lett.*, **B393** (1997) 375.
- 68. D.P.Kirilova and M.V.Chizhov, *Phys.Rev.*, **D58** (1998) 073004.
- 69. D.P.Kirilova and M.V.Chizhov, *Nucl. Phys.*, **B534** (1998) 447.
- 70. D.P.Kirilova and M.V.Chizhov, hep-ph/9908525.
- 71. D.P.Kirilova and M.V.Chizhov, hep-ph/9909408.
- 72. A. D. Dolgov, S. H. Hansen, S. Pastor, and D. V. Semikoz, hep-ph/9910444.
- 73. R.A. Harris, L. Stodolsky, *Phys. Lett.* **B116** (1992) 464; X. Shi, D.N. Schramm, and B.D. Fields, *Phys. Rev.* **D48** (1993) 2563.
- 74. N.F. Bell, R.R. Volkas, and Y.Y.Y. Wong, *Phys. Rev.* **D59** (1999) 113001.